The Definition of Square Roots

A square root of a number is a number that when multiplied by itself yields the original number. For example, 4 is a square root of 16, because \(4^2 = 16\). Since \((-4)^2 = 16\), we can say that \(-4\) is a square root of 16 as well. Every positive real number has two square roots, one positive and one negative. For this reason, we use the radical sign \(\sqrt{}\) to denote the principal (nonnegative) square root and a negative sign in front of the radical \(-\sqrt{}\) to denote the negative square root.

\[
\sqrt{16} = 4 \quad \text{Positive square root of } 16 \\
-\sqrt{16} = -4 \quad \text{Negative square root of } 16
\]

Zero is the only real number with exactly one square root. 
\[
\sqrt{0} = 0
\]

If the radicand, the number inside the radical sign, is nonzero and can be factored as the square of another nonzero number, then the square root of the number is apparent. In this case, we have the following property:

\[
\sqrt{a^2} = a, \text{ if } a \geq 0
\]

So, if we have \(\sqrt{9}\), that is the same as \(\sqrt{3^2}\), which is the same as 3. It is important to point out that \(a\) is required to be nonnegative.

Note that if \(a < 0\), then we would get something like this: \(\sqrt{(-3)^2} = \sqrt{9} = 3\), and not -3. Squaring the negative will make it positive, and the square root will keep it positive. Some will make the definition that \(\sqrt{a^2} = |a|\) since both sides keep the sign the same when \(a\) is positive, but changes the sign of \(a\) when \(a\) is negative. You will investigate this further in a later course.
Example 1

Find the square root:

a. $\sqrt{121}
\hspace{2cm} = \sqrt{11^2} = 11$

b. $\sqrt{0.25}
\hspace{2cm} = \sqrt{0.5^2} = 0.5$

c. $\sqrt{\frac{4}{9}}
\hspace{2cm} = \sqrt{\left(\frac{2}{3}\right)^2} = \frac{2}{3}$

Solution:

Example 2

Find the negative square root:

a. $-\sqrt{64}
\hspace{2cm} = -8$

b. $-\sqrt{1}
\hspace{2cm} = -1$

Solution:

The radicand may not always be a perfect square. If a positive integer is not a perfect square, then its square root will be irrational. Consider $\sqrt{5}$, we can obtain an approximation by bounding it using the perfect squares 4 and 9 as follows:

$4 < \sqrt{5} < 9$
\hspace{2cm} $2 < \sqrt{5} < 3$

With this we conclude that $\sqrt{5}$ is somewhere between 2 and 3. This number is better approximated on most calculators using the square root button, $\sqrt{}$.

*Note: Use $[2^{nd}] [ x^2 ]$ on the TI-83 or TI-84. $\sqrt{5} \approx 2.236$ because $2.236^2 \approx 5$. Note: Using more decimals gives more accurate results.*
Next, consider the square root of a negative number. To determine the square root of \(-9\), you must find a number that when squared results in \(-9\),

\[
\sqrt{-9} = ? \text{ or } (?)^2 = (?)(?)(?) = -9
\]

However, any real number squared always results in a positive number,

\[
(3)^2 = 9 \text{ and } (-3)^2 = (-3)(-3) = 9
\]

Try calculating \(\sqrt{-9}\) on your calculator; what does it say? It may say there is an error, or it may give an answer of \(3i\).

In this section, we will state that \(\sqrt{-9}\) is not a real number. The square root of a negative number is defined in future courses as an imaginary or complex number, but this topic is beyond the scope of this section. For now, we will just say the square root of a negative number is not a real number.

**The Definition of Cube Roots**

A **cube root** of a number is a number that when multiplied by itself three times yields the original number. Furthermore, we denote a cube root using the symbol \(\sqrt[3]{\phantom{0}}\), where 3 is called the index.

For example,

\[
\sqrt[3]{8} = 2, \text{ because } 2^3 = 8
\]

The product of three equal factors will be positive if the factor is positive, and negative if the factor is negative. For this reason, any real number will have only one real cube root. Hence the technicalities associated with the principal root do not apply. For example,

\[
\sqrt[3]{-8} = -2, \text{ because } (-2)^3 = -8
\]

In general, given any real number \(a\), we have the following property:

\[
\sqrt[3]{a^3} = a
\]

When simplifying cube roots, look for factors that are perfect cubes.
Example 3

Find the cube root:

a. $\sqrt[3]{125} = \sqrt[3]{5^3} = 5$

b. $\sqrt[3]{0} = \sqrt[3]{0^3} = 0$

c. $\sqrt[3]{\frac{8}{27}} = \sqrt[3]{\left(\frac{2}{3}\right)^3} = \frac{2}{3}$

Solution:

Example 4

Find the cube root:

a. $\sqrt[3]{-27} = \sqrt[3]{(-3)^3} = -3$

b. $\sqrt[3]{-1} = \sqrt[3]{(-1)^3} = -1$

Solution:

It may be the case that the radicand is not a perfect cube. If this is the case, then its cube root will be irrational. For example, $\sqrt[3]{2}$ is an irrational number, which can be approximated on most scientific calculators using the root button $\sqrt[n]{\cdot}$ or by using a power of $\frac{1}{n}$.

On the TI83-84, you can hit [Math], then use option 4 for the cube root. You can also enter your number followed by $^{(1/3)}$ to compute a cube root.

Therefore, we have $\sqrt[3]{2} \approx 1.260$, because $1.260^3 \approx 2$.

Note: To calculate $\sqrt[n]{\cdot}$ with other indices, you can use $2^{(1/n)}$ on the calculator to compute higher index roots.

It is important to point out that a square root has index 2; therefore, the following are equivalent:

$\sqrt[3]{a} = \sqrt[3]{a}$

In other words, if no index is given, it is assumed to be the square root.
Simplifying Square Roots

It will not always be the case that the radicand is a perfect square. If not, we use the following two properties to simplify the expression. Given real numbers \( \sqrt{A} \) and \( \sqrt{B} \) where \( B \neq 0 \),

| Product Rule for Radicals: \( \sqrt{A} \cdot \sqrt{B} = \sqrt{A \cdot B} \) |
| Quotient Rule for Radicals: \( \sqrt{\frac{A}{B}} = \frac{\sqrt{A}}{\sqrt{B}} \) |

A simplified radical is one where the radicand does not consist of any factors that can be written as perfect powers of the index. That means a square root, once it is simplified, will not contain any factors that are perfect squares still inside the root. A cube root, once it is simplified, will not contain any factors that are perfect cubes still inside the root.

Given a square root, the idea is to identify the largest square factor of the radicand and then apply the property shown above. As an example, to simplify \( \sqrt{12} \), notice that 12 is not a perfect square. However, 12 does have a perfect square factor of 4, \( 12 = 4 \cdot 3 \). Apply the property as follows:

\[
\sqrt{12} = \sqrt{4} \cdot \sqrt{3} \\
= \sqrt{4} \cdot \sqrt{3} \\
= 2\sqrt{3}
\]

The number \( 2\sqrt{3} \) is a simplified irrational number, and is considered an exact answer. You may be asked to find an approximate answer rounded off to a certain decimal places at the end of a problem. In that case, use a calculator to find the decimal approximation using either the non-simplified root or the simplified equivalent.

\[
\sqrt{12} = 2\sqrt{3} \approx 3.46
\]

As a check, calculate \( \sqrt{12} \) and \( 2\sqrt{3} \) on a calculator and verify that the results are both approximately 3.46.

List of Commonly Used Perfect Powers

<table>
<thead>
<tr>
<th>Squares:</th>
<th>Cubes:</th>
<th>4(^{th}) Powers:</th>
<th>5(^{th}) Powers:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2^2 = 4 )</td>
<td>( 8^2 = 64 )</td>
<td>( 2^4 = 16 )</td>
<td>( 2^5 = 32 )</td>
</tr>
<tr>
<td>( 3^2 = 9 )</td>
<td>( 9^2 = 81 )</td>
<td>( 3^4 = 81 )</td>
<td>( 3^5 = 243 )</td>
</tr>
<tr>
<td>( 4^2 = 16 )</td>
<td>( 10^2 = 100 )</td>
<td>( 4^4 = 256 )</td>
<td></td>
</tr>
<tr>
<td>( 5^2 = 25 )</td>
<td>( 11^2 = 121 )</td>
<td>( 5^4 = 625 )</td>
<td></td>
</tr>
<tr>
<td>( 6^2 = 36 )</td>
<td>( 12^2 = 144 )</td>
<td>( 6^4 = 216 )</td>
<td></td>
</tr>
<tr>
<td>( 7^2 = 49 )</td>
<td>( 13^2 = 169 )</td>
<td>( 7^4 = 343 )</td>
<td></td>
</tr>
</tbody>
</table>
Example 5

Simplify: $\sqrt{135}$

Solution:
Begin by finding the largest perfect square factor of 135:

$$135 = 3 \cdot 3 \cdot 3 \cdot 5$$
$$\quad = 3^2 \cdot 3 \cdot 5$$
$$\quad = 9 \cdot 15$$

Therefore,

$$\sqrt{135} = \sqrt{9 \cdot 15} \quad \text{Apply the product rule for radicals.}$$
$$\quad = \sqrt{9} \cdot \sqrt{15} \quad \text{Simplify roots of perfect squares.}$$
$$\quad = 3 \sqrt{15}$$

Answer: $3 \sqrt{15}$

Example 6

Simplify: $\frac{\sqrt{108}}{\sqrt{169}}$

Solution:
We begin by finding the prime factorizations of both 108 and 169. This will enable us to easily determine the largest perfect square factors.

$$108 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 = 2^2 \cdot 3^2 \cdot 3$$
$$169 = 13 \cdot 13 = 13^2$$

Therefore,

$$\frac{\sqrt{108}}{\sqrt{169}} = \frac{\sqrt{2^2 \cdot 3^2 \cdot 3}}{\sqrt{13^2}} \quad \text{Apply product and quotient rule for radicals.}$$
$$\quad = \frac{\sqrt{2^2} \cdot \sqrt{3^2} \cdot \sqrt{3}}{\sqrt{13^2}} \quad \text{Simplify square roots of perfect squares.}$$
$$\quad = \frac{2 \cdot 3 \sqrt{3}}{13}$$
$$\quad = \frac{6 \sqrt{3}}{13}$$

Answer: $\frac{6 \sqrt{3}}{13}$
Example 7 shows how you can write your work when you recognize perfect squares like those given in the “List of Commonly Used Perfect Powers” on page 5. If you recognize that 81 is the same as $9^2$, then when you write $\sqrt{81}$ in your work, you can simplify it to 9 in the next step.

**Example 7**

Simplify: $-5\sqrt{162}$

$-5\sqrt{162} = -5 \cdot \sqrt{81 \cdot 2}$

Solution:

$= -5 \cdot \sqrt{81} \cdot \sqrt{2}$

$= -5 \cdot 9 \cdot \sqrt{2}$

$= -45\sqrt{2}$

Answer: $-45\sqrt{2}$

**Try this!**

Simplify: $4\sqrt{150}$.

Answer: $20\sqrt{6}$

**Roots and Exponents – Inverse Operations**

One of the main ideas to see with exponents and roots is that they are inverse operations. Consider the following calculations:

$\sqrt{9^2} = 9$

$(\sqrt{9})^2 = 9$

When you take the square root of a positive number squared, you get the same number. When you square the square root of a positive number, you get the same number. Notice that it does not matter if we apply the exponent first or the square root first. This is true for any positive real number. We can then say the following for square roots:

$\sqrt{a^2} = (\sqrt{a})^2 = a$ when $a \geq 0$

Because the square and the square root “undo each other”, we can say that they are inverse operations for positive real numbers.
Notice the same thing is true for cubes and cube roots:

\[ \sqrt[3]{8^3} = 8 \quad \text{and} \quad \sqrt[3]{(-8)^3} = -8 \]

In this example, the cube and cube root are inverse operations, and we can make the following statement about cube roots:

\[ \sqrt[3]{a^3} = \left(\sqrt[3]{a}\right)^3 = a \]

Notice that with cube roots, this works for negative numbers as well since the cube roots of negative numbers are still real numbers.

**Example 10**

Simplify: \( (\sqrt[10]{10})^3 \)

Solution:

Apply the fact that \( \sqrt[3]{a} = a \).

\( (\sqrt[10]{10})^3 = 10 \)

**Pythagorean Theorem**

There is another way to think about square roots, and that is in a geometric context. If you look at the figure shown, the area of the big box is 4, and half of the area is shaded, so the area of the shaded portion is 2. Since it is a square, the area is given by the formula \( side \cdot side \), or \( (side)^2 \). This means the length of the sides of the shaded region is \( \sqrt{2} \), since \( Area = (side)^2 = (\sqrt{2})^2 = 2 \).
A famous theorem that allows us to visualize other square roots is the Pythagorean Theorem. A **right triangle** is a triangle where one of the angles measures 90°. The side opposite the right angle is the longest side, called the **hypotenuse**, and the other two sides are called **legs**. Numerous real-world applications involve this geometric figure. The Pythagorean theorem states that for any right triangle with legs measuring \(a\) and \(b\) units, the square of the measure of the hypotenuse \(c\) is equal to the sum of the squares of the measures of the legs, or \(a^2 + b^2 = c^2\).

In other words, the hypotenuse of any right triangle is equal to the square root of the sum of the squares of its legs.

\[
\begin{align*}
  a^2 + b^2 &= c^2 \\
  c &= \sqrt{a^2 + b^2}
\end{align*}
\]

**Example 11**

Calculate the diagonal of a rectangle with sides measuring 2 units and 6 units.

**Solution:**
The diagonal of a rectangle will form a right triangle with legs 2 and 6 units long.

\[
\begin{align*}
  c^2 &= 2^2 + 6^2 = 4 + 36 = 40 \\
  c &= \sqrt{40}
\end{align*}
\]

If \(c^2 = 40\), then \(c = \sqrt{40}\)

**Simplify:**
\[
  c = \sqrt{40} = \sqrt{4 \cdot 10} = 2 \cdot \sqrt{10}
\]

**Answer:** 2\(\sqrt{10}\) units
KEY TAKEAWAYS

- The square root of a number is a number that when squared results in the original number. The principal square root of a positive real number is the positive square root. The square root of a negative number is currently left undefined.
- When simplifying the square root of a number, look for perfect square factors of the radicand. Apply the product or quotient rule for radicals and then simplify.
- The cube root of a number is a number that when cubed results in the original number. Every real number has only one real cube root.
- When simplifying cube roots, look for perfect cube factors of the radicand. Apply the product or quotient rule for radicals and then simplify.
- Exponents and roots with a matching index are inverse operations.
- The Pythagorean theorem gives us a property of right triangles: \( a^2 + b^2 = c^2 \) and \( c = \sqrt{a^2 + b^2} \) when \( a \) and \( b \) represent the lengths of the legs of a right triangle and \( c \) represents the length of the hypotenuse of the right triangle.

TOPIC EXERCISES

PART A: SQUARE AND CUBE ROOTS

Simplify.

1. \( \sqrt{81} \)
2. \( \sqrt{49} \)
3. \( -\sqrt{16} \)
4. \( -\sqrt{100} \)
5. \( \sqrt[2]{\frac{25}{16}} \)
6. \( \sqrt[2]{\frac{9}{64}} \)
7. \( \sqrt[2]{\frac{1}{4}} \)
8. \( \sqrt[2]{\frac{1}{100}} \)
9. \( \sqrt{-1} \)
10. \( \sqrt{-25} \)
11. \( \sqrt{0.36} \)
12. \( \sqrt{1.21} \)
13. \( \sqrt{(-5)^2} \)
14. \( \sqrt{(-6)^2} \)
15. \( 2\sqrt{64} \)
16. \( 3\sqrt{36} \)
17. \( -10\sqrt{4} \)
18. \( -8\sqrt{25} \)
19. \( \frac{3}{64} \)
20. \( \frac{3}{125} \)
21. \( \frac{3}{-27} \)
22. \( \frac{3}{-1} \)
23. \( \frac{3}{0} \)
24. \( \frac{3}{0.008} \)
25. \( \frac{3}{0.064} \)
26. \( -\frac{3}{-8} \)
27. \( -\frac{3}{1000} \)
28. \( \frac{3}{(-8)^3} \)
29. \( \frac{3}{(-15)^3} \)
30. \( \frac{3}{\frac{1}{216}} \)
31. \(\frac{\sqrt[3]{27}}{\sqrt[3]{64}}\)  
34. \(5 \sqrt[3]{343}\)  
35. \(4 \sqrt[3]{512}\)  
36. \(-10 \sqrt[3]{8}\)  
37. \(-6 \sqrt[3]{-64}\)  
38. \(8 \sqrt[3]{-8}\)

Use a calculator to approximate to the nearest hundredth.

39. \(\sqrt[3]{3}\)  
40. \(\sqrt[3]{10}\)  
41. \(\sqrt[3]{19}\)  
42. \(\sqrt[3]{7}\)  
43. \(3\sqrt[3]{5}\)  
44. \(-2\sqrt[3]{3}\)  
45. \(\sqrt[3]{3}\)  
46. \(\sqrt[3]{6}\)  
47. \(\frac{1}{2}\sqrt[3]{28}\)  
48. \(\sqrt[3]{9}\)  
49. \(4\sqrt[3]{3}\)  
50. \(-3 \sqrt[3]{12}\)

51. Find the squares of the first twelve positive integers.  
52. Find the cubes of the first twelve positive integers.

**PART B: SIMPLIFYING SQUARE ROOTS**

Simplify.

53. \(\sqrt{18}\)  
54. \(\sqrt{50}\)  
55. \(\sqrt{24}\)  
56. \(\sqrt{40}\)  
57. \(\sqrt{\frac{50}{81}}\)  
58. \(\sqrt{\frac{54}{25}}\)  
59. \(4\sqrt{72}\)  
60. \(3\sqrt{27}\)  
61. \(-5\sqrt{80}\)  
62. \(-6\sqrt{128}\)  
63. \(3\sqrt{-40}\)  
64. \(5\sqrt{-160}\)  
65. \((\sqrt{64})^2\)  
66. \((\sqrt{25})^2\)  
67. \((\sqrt{2})^2\)  
68. \((\sqrt{6})^2\)

**PART C: PYTHAGOREAN THEOREM**

69. If the two legs of a right triangle measure 3 units and 4 units, then find the length of the hypotenuse.  
70. If the two legs of a right triangle measure 6 units and 8 units, then find the length of the hypotenuse.
71. If the two equal legs of an isosceles right triangle measure 7 units, then find the length of the hypotenuse.
72. If the two equal legs of an isosceles right triangle measure 10 units, then find the length of the hypotenuse.
73. Calculate the diagonal of a square with sides measuring 3 centimeters.
74. Calculate the diagonal of a square with sides measuring 10 centimeters.
75. Calculate the diagonal of a square with sides measuring $\sqrt{6}$ centimeters.
76. Calculate the diagonal of a square with sides measuring $\sqrt{10}$ centimeters.
77. Calculate the length of the diagonal of a rectangle with dimensions 4 centimeters by 8 centimeters.
78. Calculate the length of the diagonal of a rectangle with dimensions 8 meters by 10 meters.
79. Calculate the length of the diagonal of a rectangle with dimensions $\sqrt{3}$ meters by 2 meters.
80. Calculate the length of the diagonal of a rectangle with dimensions $\sqrt{6}$ meters by $\sqrt{10}$ meters.
81. To ensure that a newly built gate is square, the measured diagonal must match the distance calculated using the Pythagorean theorem. If the gate measures 4 feet by 4 feet, what must the diagonal measure in inches? (Round off to the nearest tenth of an inch.)
82. If a doorframe measures 3.5 feet by 6.6 feet, what must the diagonal measure to ensure that the frame is a perfect rectangle?

**PART D: DISCUSS**

83. What does your calculator say after taking the square root of a negative number? Share your results with others and explain why it says that.
ANSWERS

1. 9
2. −4
3. \( \frac{5}{4} \)
4. \( \frac{1}{2} \)
5. Not a real number.
6. 0.6
7. 5
8. 16
9. −20
10. 4
11. −3
12. 0
13. 0.4
14. −10
15. −15
16. \( \frac{3}{4} \)
17. −\( \frac{1}{3} \)
18. 32
19. 24
20. 1.73
21. 4.36
22. 6.71
23. 1.44
24. 3.04
25. 5.77
26. 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144
27. 3\( \sqrt{2} \)
28. 2\( \sqrt{6} \)
29. \( \frac{5\sqrt{2}}{9} \)
30. 24\( \sqrt{2} \)
31. −20\( \sqrt{5} \)
32. Not a real number.
33. 64
34. 2
35. 5 units
36. 7\( \sqrt{2} \) units
37. 3\( \sqrt{2} \) centimeters
38. 2\( \sqrt{3} \) centimeters
39. 4\( \sqrt{5} \) centimeters
40. \( \sqrt{7} \) meters
41. The diagonal must measure approximately 67.9 inches.
42. Answers will vary. Either an error in the calculator, or a number times \( i \), depending on the calculator mode.